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AN ITERATIVE ALGORITHM FOR LIMIT ANALYSIS WITH NONLINEAR YIELD FUNCTIONS

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Abstract—A mathematical programming algorithm is proposed for the general limit analysis problem. Plastic behavior is described by a set of linear or nonlinear yield functions. Abstract formulations of limit analysis are first considered and a Quasi-Newton strategy for solving the optimality conditions is then sketched. The structure of the problem arising from a finite element discretization is taken into account in order that the algorithm should be able to solve large scale problems.

1. INTRODUCTION

The aim of this work is to present a new mathematical programming algorithm specially designed for the general limit analysis problem. In order to develop a method with practical applicability, we consider the structure of the problem arising from a large scale discretization of a body whose plastic behavior is described by a set of linear or nonlinear yield functions. Associated plasticity is assumed and hardening/softening effects are neglected.

Limit Analysis deals with the determination of loads producing plastic collapse, that is, a state where purely plastic strain rates take place under constant stress distribution [see e.g. Cohn and Maier (1977) and Feijóo and Zouain (1987)].

The variational characterization of plastic collapse, and the extremum principles for limit analysis of continuum bodies under proportional loads, are briefly presented in the first section of this paper. Then, duality concepts are used to show that a single numerical method is able to deal with the limit load computations coming from kinematical, equilibrium or mixed discretizations.

Linear Programming has been widely used in limit analysis because it is commonly accepted that this approach allows the solution of large scale problems [see e.g. Cohn and Maier (1977), Cristiansen (1981), Feijóo and Zouain (1987) and Borges *et al.* (1989, 1990)]. In most cases, the transformation of the continuum variational formulation into a linear programming problem involves the approximation of the yield surface by a polyhedron, besides the selection of points where plastic admissibility is exactly enforced.

We explore in this paper the nonlinear programming approach, in which no linearization of the yield criteria is made. Consequently, we expect to deal with a number of nonlinear constraints much less than the number of constraints used in the linear programming approach.

It is worth noting that, even for the general linear programming problem, there are now some iterative algorithms which look much more promising than the finite termination methods like the Simplex algorithm. This fact suggests that a single iterative procedure should be used for the nonlinear formulations of limit analysis, and also for the particular case when the exact formulation is indeed a linear programming problem.

The main idea for the development of the present method is to identify just the formal structure of the discrete limit analysis problem, including the general features related to the finite element discretizations, and then apply a mathematical programming technique to generate the solving algorithm. This basic mathematical programming technique has been proposed by Herskovits (1986) and Herskovits and Coelho (1989) and is modified here. It operates by performing a quasi-Newton iteration on the set of equalities in the optimality conditions, followed by a small deflection in order to obtain a new direction which is feasible with respect to the inequality conditions. In the application of these guidelines to limit analysis a new approach is adopted for the treatment of the equality constraints and also a special sequence of substitutions is used at the quasi-Newton stage.

2. LIMIT ANALYSIS OF A CONTINUUM

This section contains a brief description of the continuum formulations for limit analysis. The basic notations for kinematics, equilibrium and the constitutive relation are introduced in what follows.

Consider a body occupying an open bounded region $\mathscr{B}$ with regular boundary Γ. We investigate the onset of unbounded deformations from this known reference configuration, the so-called incipient plastic collapse.

Let V denote the function space of all admissible velocity fields v complying with homogeneous boundary conditions prescribed on a part Γ_u of Γ. The strain rate tensor fields D are elements of the function space W, and the tangent deformation linear operator $\mathscr{D}$ maps V into W.

Accordingly, the stress fields T belong to the function space W', which is the dual of W, and the duality product is written

$$\langle T, D\rangle = \int_{\mathscr{B}} T\cdot \mathrm{D}\,\mathrm{d}\mathscr{B}. \tag{1}$$

This expression gives the internal power of the stress T due to the strain rate field D.

Any load system is represented by an element F of the space V', the dual of V, and the corresponding duality product is denoted

$$\langle F, v\rangle = \int_{\mathscr{B}} b\cdot v\,\mathrm{d}\mathscr{B} + \int_{\Gamma_\tau} \tau\cdot v\,\mathrm{d}\Gamma, \tag{2}$$

where b and τ are body and surface loads respectively, and Γ_τ the region of Γ where tractions are prescribed ($\Gamma = \Gamma_u \cup \Gamma_\tau$ and $\Gamma_u \cap \Gamma_\tau$ is empty). The above expression gives the external power of the force system (b, τ) associated with the velocity field v.

The equilibrium condition, relating a stress field $T \in W'$ and a load system $F \in V'$ is imposed by the Principle of Virtual Power:

$$\langle T, \mathscr{D}v\rangle = \langle F, v\rangle \quad \forall v \in V, \tag{3}$$

that is, the stress $T \in W'$ which equilibrates $F \in V'$ is such that the internal power equals the external power for any kinematically admissive velocity field $v \in V$.

Equivalently, we write the above relation in compact form as

$$T \in S(F), \tag{4}$$

where $S(F)$ is the set of all stress fields in equilibrium with the given system of forces F.

We write next the constitutive relations describing a material which behaves as ideally plastic when stresses fulfill the yield criterion, i.e. hardening/softening effects are neglected. The principle of maximum dissipation is assumed to hold so that the plastic strain rate will follow the associative normality law.

The stress field T in an elastic–ideally plastic body $\mathscr{B}$ is constrained to fulfill the plastic admissibility condition, i.e. it must belong to the set

$$P = \{T \in W' \mid f(T) \leqslant 0 \quad \text{in } \mathscr{B}\}, \tag{5}$$

where f is a vector valued function describing the yield criterion. The inequality above is then understood as imposing that each component f_k, which is a regular convex function of T, is nonpositive.

The constitutive relations can be derived from the principle of maximum dissipation, which associates a stress T, and a plastic dissipation $X(D^{\mathrm{p}})$ to a given plastic strain rate D^{p} by means of

$$X(D^{\mathrm{p}}) = \sup_{T^* \in P} \langle T^*, D^{\mathrm{p}} \rangle. \tag{6}$$

Stress and plastic strain rates are thus related by the following optimality conditions of the above problem:

$$D^{\mathrm{p}} = f_{\mathrm{T}}(T)\dot{\lambda}, \tag{7}$$

$$\dot{\lambda} \in \Lambda^+, \tag{8}$$

$$\langle f(T), \dot{\lambda}^* - \dot{\lambda} \rangle \leqslant 0 \quad \forall \dot{\lambda}^* \in \Lambda^+, \tag{9}$$

where f_{T} denotes the gradient of f, $\dot{\lambda}$ is the plastic multiplier vector field whose components correspond to each plastic mode in f, Λ^+ is the set of fields with non-negative value in any point of the body. The set Λ^+ is imbeded in a space Λ endowed with the duality product

$$\langle f, \dot{\lambda} \rangle = \int_{\mathscr{B}} f \cdot \dot{\lambda} \, \mathrm{d}\mathscr{B}. \tag{10}$$

The present constitutive relation can be cast in a compact form by using the concept of subdifferential [see e.g. Panagiotopoulos (1985), Romano and Sacco (1985), Eve *et al.* (1989) and Borges *et al.* (1989)]. In fact, the relation enforced by (6) is equivalent to

$$T \in \partial X(D^{\mathrm{p}}), \tag{11}$$

where the subdifferential $\partial X(D^{\mathrm{p}})$ is the set of all stress fields such that

$$X(D^{\mathrm{p}^*}) - X(D^{\mathrm{p}}) \geqslant \langle T, D^{\mathrm{p}^*} - D^{\mathrm{p}} \rangle \quad \forall D^{\mathrm{p}^*} \in W. \tag{12}$$

Under the assumption of proportional loading, the limit analysis problem consists of finding a load factor α such that the body undergoes plastic collapse when subject to the reference loads F uniformly amplified by α. In turn, a system of loads produces plastic collapse if there exists a stress field in equilibrium with these loads, which is plastically admissible and related, by the constitutive equations, to a plastic strain rate field which is kinematically admissible.

Thus, the limit analysis problem consists of finding $\alpha \in \mathbb{R}$, $T \in W'$, $D^{\mathrm{p}} \in W$ and $v \in V$ such that:

$$D^{\mathrm{p}} = \mathscr{D}v, \quad v \in V, \tag{13}$$

$$T \in S(\alpha F), \tag{14}$$

$$T \in \partial X(D^{\mathrm{p}}). \tag{15}$$

We define in what follows the classical extremum principles of limit analysis, which can be derived from the optimality conditions above (Romano and Sacco, 1985; Eve *et al.*, 1989):

(i) Static formulation

$$\alpha = \sup_{\substack{\alpha^* \in R \\ T \in W'}} \alpha^* \mid T \in P \cap S(\alpha F); \tag{16}$$

(ii) Mixed formulation

$$\alpha = \inf_{v \in V} \sup_{T \in W'} \langle T, \mathscr{D}v \rangle \left| \begin{array}{l} \langle F, v \rangle = 1, \\ T \in P; \end{array} \right. \tag{17}$$

(iii) Kinematic formulation

$$\alpha = \inf_{v \in V} X(\mathscr{D}v) \mid \langle F, v \rangle = 1. \tag{18}$$

3. THE DISCRETE LIMIT ANALYSIS PROBLEM

All discretized versions of the limit analysis formulations (16), (17) or (18) lead to a single type of finite dimensional problem, which can be cast in four strictly equivalent forms, namely the static, mixed and kinematic discrete formulations, and the set of discrete optimality conditions. For instance, a particular finite element discretization of the kinematic principle (18) gives rise to a discrete model which can be stated in four dual forms, all having exactly the same solution. In this case, the discrete model is called kinematical. In Section 5, we describe a kinematic model for plane problems.

Then, the discrete limit analysis problem consists of finding a load factor $\alpha \in \mathbb{R}$, a stress vector $T \in \mathbb{R}^q$, a velocity vector $v \in \mathbb{R}^n$ and a plastic multiplier vector $\lambda \in \mathbb{R}^m$, such that the system represented by a deformation matrix $B: \mathbb{R}^n \to \mathbb{R}^q$ and a convex function $f(T) \in \mathbb{R}^m$, undergoes plastic collapse for some load being proportional to a given force vector $F \in \mathbb{R}^n$.

It is assumed that all rigid motions are ruled out by the kinematic constraints, so that the kernel of matrix B only contains the null velocity vector.

The four formulations below are equivalent statements of the discrete limit analysis problem in view of the convexity of $f(T)$ [see e.g. Cohn and Maier (1977), Christiansen (1981), Feijóo and Zouain (1987) and Borges *et al.* (1989, 1990)].

(i) Static formulation

$$\alpha = \max_{\substack{\alpha^* \in R \\ T \in R^q}} \alpha^* \left| \begin{array}{l} B^{\mathrm{T}} T - \alpha^* F = 0, \\ f(T) \leqslant 0; \end{array} \right. \tag{19}$$

(ii) Mixed formulation

$$\alpha = \min_{v \in R^n} \max_{T \in R^q} T \cdot Bv \left| \begin{array}{l} F \cdot v = 1, \\ f(T) \leqslant 0; \end{array} \right. \tag{20}$$

(iii) Kinematic formulation

$$\alpha = \min_{v \in R^n} X(Bv) \mid F \cdot v = 1, \tag{21}$$

where

$$X(Bv) = \max_{T \in R^q} T \cdot Bv \mid f(T) \leqslant 0; \tag{22}$$

(iv) Optimality conditions

$$Bv - f_T(T)\dot{\lambda} = 0, \tag{23}$$

$$B^T T - \alpha F = 0, \tag{24}$$

$$F \cdot v = 1, \tag{25}$$

$$f_j(T)\dot{\lambda}_j = 0, \quad j = 1, \ldots, m, \tag{26}$$

$$f(T) \leqslant 0, \tag{27}$$

$$\dot{\lambda} \geqslant 0. \tag{28}$$

4. A PROCEDURE TO SOLVE THE DISCRETE LIMIT ANALYSIS PROBLEM

In this section we propose a general iterative algorithm to solve the discrete limit analysis problem by solving the discrete optimality conditions (23)–(28).

The present approach differs then to direct minimization techniques based on the static or kinematic forms. For instance, the one proposed by Huh and Yang (1991) avoids the nondifferentiability of the kinematic form by combining smoothing and successive approximation.

Subsections 4.1–4.6 contain the complete development of the algorithm, which is summarized in subsection 4.7. Obviously, all the operations shown in the explanation of the algorithm are not necessary for the implementation described in the final subsection.

The algorithm consists, schematically, of the use of a quasi-Newton iteration formula, associated with the set of all equalities in the optimality conditions, followed by a small deflection in order to preserve feasibility with respect to the inequality conditions.

The set of optimality conditions (23)–(28) can be cast in the form:

$$\Psi(x) = 0, \quad f(T) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \tag{29}$$

where

$$x = [T, v, \alpha, \dot{\lambda}]^T, \tag{30}$$

$$\Psi(x) = \begin{bmatrix} Bv - f_T(T)\dot{\lambda} \\ B^T T - \alpha F \\ -F \cdot v + 1 \\ -G(T)\dot{\lambda} \end{bmatrix}, \tag{31}$$

$$G(T) = \operatorname{diag}(f_j(T)). \tag{32}$$

In the following subsections we describe the procedure to find a new iterate $\bar{x}$ from the present value x, by defining a search direction d_x and a step length s such that

$$\bar{x} = x + s d_x. \tag{33}$$

The search direction d_x is determined in a two-stage process. In the first stage, described in subsection 4.1, we define an increment estimate d_x^0 by performing a quasi-Newton iteration for the nonlinear equations in (29). This involves the solution of a system of linear equations. In the second stage, explained in subsection 4.2, the increment estimate d_x^0 is slightly deflected to obtain a search direction d_x which is feasible with respect to the inequalities in (29). This is achieved by solving once again the same linear system, with a perturbation, depending on d_x^0, in the right-hand side.

4.1. *Increment estimate*

An estimate d_x^0 of the increment in x is computed using the following iteration formula related to $\Psi(x) = 0$:

$$\mathscr{J}(x)d_x^0 = -\Psi(x), \tag{34}$$

where

$$\mathscr{J}(x) = \nabla\Psi(x), \tag{35}$$

if we decide to use Newton's formula. In this case

$$\mathscr{J}(x) = \begin{bmatrix} -H & B & 0 & -f_T \\ B^T & 0 & -F & 0 \\ 0 & -F^T & 0 & 0 \\ -\Lambda f_T^T & 0 & 0 & -G \end{bmatrix}, \tag{36}$$

$$\Lambda = \mathrm{diag}\,(\dot{\lambda}_j), \tag{37}$$

$$H = \Sigma\dot{\lambda}_j\nabla^2 f_j. \tag{38}$$

However, we can choose some fixed or updated matrix $\tilde{C}_j$ in place of $\nabla^2 f_j$ in order to avoid the Hessian computation. Consequently, we adopt the general form (36) for the iteration matrix, and assume that H is updated by any prescribed rule preserving symmetry and positive definiteness.

The iteration formula (34) is expanded next, by using (31) and (36), assuming that the equilibrium constraint (24) is exactly satisfied for the present T and α. This leads to

$$Hd_T^0 - Bv^0 + f_T\dot{\lambda}^0 = 0, \tag{39}$$

$$B^T d_T^0 - Fd_\alpha^0 = 0, \tag{40}$$

$$F\cdot v^0 = 1, \tag{41}$$

$$\Lambda f_T^T d_T^0 + G\dot{\lambda}^0 = 0, \tag{42}$$

where d_T^0 and d_α^0 are increment estimates for T and α, while v^0 and $\dot{\lambda}^0$ are new estimates for v and $\dot{\lambda}$ respectively.

This system of equations, written in terms of the set of unknowns d_T^0, v^0, d_α^0 and $\dot{\lambda}^0$, must be solved in order to compute the increment estimate d_x^0.

Notice that Hd_T^0 equals the difference of total and plastic strain rates in (39), so that H may be interpreted as an elastic compliance matrix, depending on T and $\dot{\lambda}$. Moreover, eqn (40) represents incremental equilibrium, while (41) imposes that the external power equals one. The meaning of eqn (42) is discussed in subsection 4.2.

There are some common features in the structure of matrices H, B, f_T and vector F that are obtained in a finite element discretization of any limit analysis formulation. For instance, the compliance matrix H is composed of disjoint blocks if the stress or strain parameters are uncoupled, that is, if each global parameter is associated to a single finite element. This is the case in a stress interpolation, when the traction acting on inter-element boundaries is made continuous by means of equilibrium constraints rather than sharing stress parameters.

In view of the above arguments, we shall use a particular sequence of substitutions for the solution of the system (39)–(42). First, we obtain from (39):

$$d_T^0 = H^{-1}(Bv^0 - f_T\dot{\lambda}^0). \tag{43}$$

This equation, and several of the following ones, only involve quantities and operations concerning one finite element at a time. For the sake of simplicity, we are omitting, whenever possible, the index denoting element.

We make the assumption that $\dot{\lambda}_j$ is strictly positive for any plastic mode j. This will be guaranteed by the updating rule performed at the end of the iteration. Then, we can substitute (43) in (42), multiplied by Λ^{-1}, to obtain

$$(f_T^T H^{-1} f_T - \Lambda^{-1} G)\dot{\lambda}^0 = f_T^T H^{-1} Bv^0. \tag{44}$$

Consequently, we define

$$W = f_T^T H^{-1} f_T - \Lambda^{-1} G \tag{45}$$

and

$$Q = H^{-1} f_T. \tag{46}$$

It is proven in the Appendix that the symmetric matrix W is positive definite under some assumptions on the plastic function f which can be physically interpreted. Hence, it follows from (44), (45) and (46) that

$$\dot{\lambda}^0 = W^{-1} Q^T Bv^0. \tag{47}$$

Substitution of this equation in (43) leads to

$$d_T^0 = \mathbb{D}^{ep} Bv^0, \tag{48}$$

where

$$\mathbb{D}^{ep} = H^{-1} - QW^{-1}Q^T. \tag{49}$$

Obviously, (48) may be interpreted as a tangent relation between strain and stress, with $\mathbb{D}^{ep}$ being the matrix of elastic–plastic moduli. We prove in the Appendix that $\mathbb{D}^{ep}$ is positive semi-definite.

Equation (45) is now substituted in (40) to get

$$Kv^0 = d_\alpha^0 F, \tag{50}$$

where the matrix K is obtained by assembling the contribution of each element to the following matrix

$$K = B^T \mathbb{D}^{ep} B. \tag{51}$$

It represents a variable elastic–plastic stiffness.

By using (50) and (41) we deduce the following sequence to compute d_α^0 and v^0: first solve

$$K\hat{v} = F, \tag{52}$$

then use $\hat{v}$ to obtain

$$d_\alpha^0 = (F \cdot \hat{v})^{-1}, \tag{53}$$

and finally set

$$v^0 = d^0_\alpha \hat{v}. \tag{54}$$

The remaining unknowns d^0_T and $\dot{\lambda}$ are now obtained, in an element-by-element basis, by substituting the value of v^0 into (48) and (47).

4.2. *The computation of a deflected feasible direction*

We generate an interior points algorithm, that is a procedure which assures that the new iterate is feasible with respect to the inequality constraints $f(T) \leqslant 0$ and $\dot{\lambda} \geqslant 0$ provided that the present approximation is feasible. Consequently, we must compute a small deflection of d^0_x in order to obtain a feasible direction, rather than a tangent, with respect to saturated inequality constraints.

Consider the component-wise form of (42)

$$\dot{\lambda}_j f^T_{jT} d^0_T = -f_j \dot{\lambda}^0_j, \tag{55}$$

from which we deduce that

$$f_j = 0 \quad \Rightarrow \quad f^T_{jT} d^0_T = 0, \tag{56}$$

because $\dot{\lambda}_j$ is strictly positive. That is, d^0_T is tangent if f_j is active.

Likewise, a strictly feasible direction d_x will be defined by d_T, $\bar{v}$, d_α and $\bar{\lambda}$ if we impose for all active plastic modes that

$$\dot{\lambda}_j f^T_{jT} d_T = -f_j \bar{\lambda}_j - \theta, \tag{57}$$

where θ represents the amount of deflection needed to ensure that $f^T_{jT} d_T$ is negative for saturated constraints. This value of θ will be chosen to be of the same order as $\|d^0_T\|^2$, which is proportional to $(d^0_\alpha)^2$ by virtue of the equilibrium equation (40):

$$\theta = \bar{\rho}(d^0_\alpha)^2. \tag{58}$$

It can be shown from (50) and (41) that d^0_α is non-negative if the matrix K is positive semi-definite. Hence (d^0_T, d^0_α) is an ascent direction for the static form of the problem. Consequently we shall set $\bar{\rho}$ small enough in order to preserve this condition. We show afterwards that this is possible, and we give a rule to compute θ in terms of the increment estimate d^0_x.

The deflected feasible direction is then obtained as the solution of the iteration equation (34), i.e. a system like (39)–(42) with the small perturbation θ added in the right-hand side of (42), which reads as follows:

$$Hd_T - B\bar{v} + f_T\bar{\lambda} = 0, \tag{59}$$

$$B^T d_T - F d_\alpha = 0, \tag{60}$$

$$F \cdot \bar{v} = 1, \tag{61}$$

$$\Lambda f^T_T d_T + G\bar{\lambda} = -\theta e, \tag{62}$$

where e is a vector with all components equal to one.

We repeat in what follows the sequence of substitutions used to solve (39)–(42) in order to solve (59)–(62). It comes from (59) that

$$d_T = H^{-1}(B\bar{v} - f_T\bar{\lambda}). \tag{63}$$

Substituting this expression in (62) and using (45) and (46) we have

$$\bar{\lambda} = W^{-1}(Q^{\mathrm{T}}B\bar{v}+\theta\Lambda^{-1}e). \tag{64}$$

This expression is used in (63) to obtain

$$d_{\mathrm{T}} = \mathbb{D}^{\mathrm{ep}}B\bar{v}-\theta QW^{-1}\Lambda^{-1}e. \tag{65}$$

Using this result in (60) we also obtain

$$K\bar{v} = d_{\alpha}F+\Theta, \tag{66}$$

where the vector

$$\Theta = \theta B^{\mathrm{T}}QW^{-1}\Lambda^{-1}e, \tag{67}$$

represents loads associated with the perturbation θ.

Combining (66) and (61) we deduce the following sequence to compute d_{α} and $\bar{v}$: first solve

$$Kv^{\theta} = \Theta, \tag{68}$$

then use v^{θ} to obtain

$$d_{\alpha} = d_{\alpha}^{0}(1-F\cdot v^{\theta}), \tag{69}$$

and finally use these values to compute

$$\bar{v} = \frac{d_{\alpha}}{d_{\alpha}^{0}}v^{0}+v^{\theta}. \tag{70}$$

The remaining unknowns d_{T} and $\bar{\lambda}$ are now obtained by substituting the value of $\bar{v}$ into (65) and (64).

Next we prove that the value of $\bar{\rho}$, and hence θ, can be computed *a priori* in such a way that $(d_{\mathrm{T}}, d_{\alpha})$ results in an ascent direction for the static form (19) of the problem. To this end we choose a fixed parameter $\beta \in (0, 1)$ and impose the following condition:

$$d_{\alpha} = \beta d_{\alpha}^{0}. \tag{71}$$

In view of (69) this means that

$$F\cdot v^{\theta} = 1-\beta. \tag{72}$$

Due to (47), (50), (67) and (68):

$$\begin{aligned} d_{\alpha}^{0}F\cdot v^{\theta} &= d_{\alpha}^{0}F\cdot K^{-1}\Theta \\ &= v^{0}\cdot\Theta \\ &= \theta v^{0}\cdot B^{\mathrm{T}}QW^{-1}\Lambda^{-1}e \\ &= \theta\Lambda^{-1}e\cdot\bar{\lambda}^{0} \\ &= \theta\Sigma(\bar{\lambda}_{j}^{0}/\bar{\lambda}_{j}), \end{aligned} \tag{73}$$

where the summation includes all plastic modes. Hence, condition (72) reads

$$\theta\Sigma(\dot{\lambda}_j^0/\dot{\lambda}_j) = d_\alpha^0(1-\beta). \tag{74}$$

Consequently, we give the following rule for updating the deflection so as to ensure that it converges quadratically with d_T^0 and gives rise to an ascent direction satisfying (57):

$$\bar{\rho} = \frac{1-\beta}{d_\alpha^0\Sigma(\dot{\lambda}_j^0/\dot{\lambda}_j)}. \tag{75}$$

4.3. *Line search*

A step length s is determined now for the deflected direction d_T. Considering the statical form of the problem it follows that any step along (d_T, d_α) will increase the objective and also preserve the validity of the equilibrium constraint, because d_T and d_α are already equilibrated. Hence, the only requirement for the line search is the plastic admissibility of final stresses.

Then, the step length is determined by

$$s = \min_j s^j \mid f_j(T+s^j d_T) = \gamma_f f_j(T), \tag{76}$$

where $\gamma_f \in (0, 1)$ is given by a prescribed rule. This prevents any plastic function from becoming active in a single iteration, while it can approach zero in a few iterations. The parameter γ_f is forced to converge to zero by means of the rule:

$$\gamma_f = \min\,[{}^0\gamma_f, d_\alpha^0/\alpha], \tag{77}$$

where ${}^0\gamma_f \in (0, 1)$ is a given control parameter.

4.4. *Updating*

The set of variables T, v, α and $\dot{\lambda}$ must be updated in order to perform, if necessary, a new iteration.

First, we use the already computed step length to update T and α, i.e.

$$T \leftarrow T+sd_T, \tag{78}$$

$$\alpha \leftarrow \alpha+sd_\alpha. \tag{79}$$

We notice that there is no need to update the velocity vector.

Finally, we give a rule for updating $\dot{\lambda}$, taking into account that it must be strictly positive so that the matrix Λ can be inverted:

$$\dot{\lambda}_j \leftarrow \max\,(\dot{\lambda}_j^0, \gamma_\lambda \|\dot{\lambda}^0\|_\infty), \tag{80}$$

$$\gamma_\lambda = \min\left[{}^0\gamma_\lambda, \frac{\|d_T\|}{\|T\|}\right], \tag{81}$$

where ${}^0\gamma_\lambda$ is a prescribed tolerance, $\|\cdot\|$ is the Euclidean norm and $\|\cdot\|_\infty$ is the norm of the maximum absolute value of components. This rule allows $\dot{\lambda}_j$ to converge to a positive Lagrange multiplier value, if this is the case for $\dot{\lambda}_j^0$, while setting to a small positive value those parameters $\dot{\lambda}_j$ corresponding to $\dot{\lambda}_j^0$ tending to zero.

4.5. *Initialization requirements*

It has been assumed in (34) that the equilibrium equation is satisfied for the starting values of each iteration. Moreover, the computed stress and load factor increments are also related by equilibrium, as a consequence of (60). Hence, it suffices to initialize the algorithm with a pair (T, α) complying with the equilibrium condition to have equilibrated stress and load approximation all along the convergence process.

The plastic admissibility of stresses at the start of each iteration has also been assumed, and it is preserved at the end by virtue of the line search procedure.

Consequently, we choose

$$T = 0, \quad \alpha = 0, \quad \dot{\lambda}_j = -\frac{1}{f_j(0)} \tag{82}$$

as initial values for the algorithm, which are feasible with respect to plastic admissibility and equilibrium. Notice that in this way all inequality constraints are initially inactive in the first iteration, thus there is no need to perform the deflection step.

4.6. *Convergence criterion*

To test whether or not convergence has been achieved, we consider the set of optimality conditions (23)–(28) and recall that the equilibrium equation (21), the external power equality (25) and the plastic admissibility constraint (27) are enforced in the iteration procedure. Then, we use v^0 and $\dot{\lambda}^0$ to check if the present value of T complies with eqns (23), (26) and (28). So, convergence is achieved if the following criterion is satisfied:

$$\| Bv^0 - f_T \dot{\lambda}^0 \|_\infty \leqslant \varepsilon_D \| Bv^0 \|_\infty, \tag{83}$$

$$\dot{\lambda}_j^0 \geqslant -\varepsilon_\lambda \| \dot{\lambda}^0 \|_\infty, \quad j = 1, \ldots, m, \tag{84}$$

$$\dot{\lambda}_j^0 \leqslant \varepsilon_\lambda \| \dot{\lambda}^0 \|_\infty \quad \text{if} \quad f_j < \varepsilon_f f_j(0), \tag{85}$$

where the parameters ε_D, ε_λ and ε_f are prescribed tolerances.

Our final remarks will be concerned with the possibility of detecting singularity or ill-conditioning of the pseudo stiffness matrix K. We recall that it has been proven that K is only positive semi-definite, although all rigid motions have been eliminated in B. Moreover, eqn (50) must be satisfied, at the solution, for the collapse velocity and d_α equal to zero, hence K necessarily tends to become singular. The question arises whether the singularity of K is a sufficient condition for convergence. Obviously, a theoretical result for this question would be very helpful. Anyway, we need to insert a test for singularity in the decomposition of K. In the numerical applications already performed the singularity of K has never been detected before the previously mentioned convergence criterion was satisfied.

4.7. *Summary of the iterative algorithm for limit analysis*

We summarize below the proposed algorithm for solving the discrete limit analysis problem.

Whenever possible, the computations are performed on an element-by-element basis. For each finite element the plastic admissibility is considered in a finite set of prescribed points. Thus, each finite element is associated with a subset of global plastic function components which collect all plastic function components of the material for all the points used to ensure plastic admissibility.

The notation $\nabla^2 f_j(T) > 0$ means that the Hessian of this plastic mode is a positive definite matrix.

(I) *Initialization*

$T = 0$
$\alpha = 0$
for each plastic mode

$$\dot{\lambda}_j = -\frac{1}{f_j(0)}$$

endfor

(II) *Increment estimate*

for each element i
for each plastic mode in the element
if (NEWTON = TRUE) and ($\nabla^2 f_j(T) > 0$)
then $\tilde{C}^j = \nabla^2 f_j(T)$
else $\tilde{C}^j = \bar{C}^j$ (prescribed)
endif
endfor
$H^{-1} = (\Sigma \dot{\lambda}_j \tilde{C}^j)^{-1}$
$Q = H^{-1} f_{\mathrm{T}}$
$W = Q^{\mathrm{T}} f_{\mathrm{T}} - \Lambda^{-1} G$
$\mathbb{D}^{\mathrm{ep}} = H^{-1} - Q W^{-1} Q^{\mathrm{T}}$
$K^i = B^{\mathrm{T}} \mathbb{D}^{\mathrm{ep}} B$
Mount K^i in K
endfor

Decompose K
if some diagonal entry of K becomes less or equal zero
then terminate
end if

Solve $K\hat{v} = F$
$d_\alpha^0 = (F \cdot \hat{v})^{-1}$
$v^0 = d_\alpha^0 \hat{v}$
for each element
$\dot{\lambda}^0 = W^{-1} Q^{\mathrm{T}} B v^0$
endfor

(III) *Convergence check*

Compute $\|\dot{\lambda}\|_\infty$, $\|Bv\|_\infty$ and $\|Bv^0 - f_{\mathrm{T}} \dot{\lambda}^0\|_\infty$
if $\|Bv^0 - f_{\mathrm{T}} \dot{\lambda}^0\|_\infty > \varepsilon_{\mathrm{D}} \|Bv^0\|_\infty$
then convergence is not achieved
go to next block
else
for each element
for each plastic mode in the element
if $\dot{\lambda}_j^0 < -\varepsilon_\lambda \|\dot{\lambda}^0\|_\infty$
then convergence is not achieved
go to next block
else
if $\dot{\lambda}_j^0 > \varepsilon_\lambda \|\dot{\lambda}^0\|_\infty$ and
$f_j < \varepsilon_{\mathrm{f}} f_j(0)$
then convergence is not achieved
go to next block
endif
endif
endfor
endfor
endif
terminate with convergence achieved

(IV) *Deflection*

$$\theta = \frac{(1-\beta) d_\alpha^0}{\Sigma(\dot{\lambda}_j^0 / \dot{\lambda}_j)}$$

for each element
$\Theta = \theta B^{T} Q W^{-1} \Lambda^{-1} e$
endfor
Solve $Kv^{\theta} = \Theta$
$d_{\alpha} = \beta d_{\alpha}^{0}$
$\bar{v} = \beta v^{0} + v^{\theta}$
for each element
$d_{T} = \mathbb{D}^{ep} B \bar{v} - \theta Q W^{-1} \Lambda^{-1} e$
endfor

(V) *Line search*

$\gamma_{f} = \min [{}^{0}\gamma_{f}, d_{\alpha}^{0}/\alpha]$
for each plastic mode
Find s^{j} such that
$f_{j}(T + s^{j} d_{T}) = \gamma_{f} f_{j}(T)$
endfor
$s = \min s^{j}$

(VI) *Updating*

$\alpha \leftarrow \alpha + s d_{\alpha}$
for each element
$T \leftarrow T + s d_{T}$
endfor

$$\gamma_{\lambda} = \min \left[{}^{0}\gamma_{\lambda}, \frac{\| d_{T} \|}{\| T \|} \right]$$

for each element
for each plastic mode
$\dot{\lambda}_{j} \leftarrow \max (\dot{\lambda}_{j}^{0}, \gamma_{\lambda} \| \dot{\lambda}^{0} \|_{\infty})$
endfor
endfor

5. APPLICATION TO PLANE STRESS AND PLANE STRAIN PROBLEMS

5.1. *Plane stress and plane strain models*

The kinematics and equilibrium for a body in plane stress or plane strain are described by means of the vector fields:

$$v = [v_{x}, v_{y}]^{T},$$
$$D = [D_{x}, D_{y}, 2D_{xy}]^{T},$$
$$T = [T_{x}, T_{y}, T_{xy}]^{T}. \tag{86}$$

The deformation operator for these problems is given by

$$\mathscr{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}. \tag{87}$$

It is considered that the von Mises or the Tresca criteria define the plastic function f. Both functions can be cast in the form:

Table 1. Matrix C for the quadratic plastic function of eqn (89)

Plane stress			Plane strain					
von Mises			Tresca			von Mises		
2	−1	0	2	−2	0	3/2	−3/2	0
−1	2	0	−2	2	0	−3/2	3/2	0
0	0	6	0	0	8	0	0	6

$$f(T) = \sigma_{eq}^2 - \sigma_Y^2, \tag{88}$$

where σ_{eq} is the equivalent stress defined by

$$\sigma_{eq} = \sqrt{\tfrac{1}{2}CT\cdot T} \tag{89}$$

and σ_Y is the material yield limit in pure traction. The matrix C is given in Table 1.

5.2. *Discretization of the two-dimensional domain*

Problems (16), (17) and (18) are defined in spaces of infinite dimension. The discrete versions of these problems are obtained by the substitution of those domains by finite dimensional function spaces.

5.2.1. *The constant deformation triangle.* A three nodes interpolation is used to generate the FEM basis for velocities, and then the kinematic minimum principle is solved in the restricted domain of piecewise linear functions. The approximation space is contained in the original domain and we shall compute exactly the integrals appearing in the objective function and the constraint. Hence, we look for an approximation α of the true collapse factor $\hat{\alpha}$, such that

$$\alpha \geqslant \hat{\alpha}. \tag{90}$$

Denoting by $N^i(x)$ the interpolation matrix, $\hat{B}^i$ the constant matrix of deformation for the ith triangle, and by χ the specific dissipation, the objective functional, restricted to the approximation functions, becomes:

$$X = \sum_i \int_{\mathscr{B}^i} \chi[\mathscr{D}N^i(x)v]\,\mathrm{d}\mathscr{B}^i = \sum_i \chi(\hat{B}^i v)\mathscr{B}^i. \tag{91}$$

Besides, substitution of the interpolation in the equality constraint leads to the following expressions of equivalent nodal forces of the ith element:

$$F^i = \int_{\mathscr{B}^i} N^{iT} b\,\mathrm{d}\mathscr{B}^i + \int_{\partial\mathscr{B}^i} N^{iT}\tau\,\mathrm{d}\Gamma, \tag{92}$$

where b are body forces and τ are surface forces.

The discretization of the kinematic principle is now complete but we can identify the dual discrete forms by substituting in (91) the expression for the specific dissipation:

$$\chi(\hat{B}^i v) = \max_{T^i} T^i \cdot \hat{B}^i v \mid f(T^i) \leqslant 0. \tag{93}$$

The dual variable T^i is now interpreted as the stress in element i.

Finally, the global matrix B is obtained by assembling

$$B^i = \hat{B}^i \mathscr{B}^i \tag{94}$$

and the global vector F by assembling F^i. Each global constraint f_j corresponds to one of the plastic modes of the yield function of the material when its arguments are substituted by the stress components of an element.

It is worth noting that this kinematic discretization procedure cannot be generalized for other types of interpolations, so the usefulness of mixed formulations becomes apparent.

5.3. *Numerical applications*

Some simple problems have been used to demonstrate the applicability of the algorithm to medium sized models.

The values adopted for control parameters of the algorithm are:

$$^0\gamma_f = 0.01, \quad ^0\gamma_\lambda = 0.1, \quad \beta = 0.7.$$

For the purpose of checking the precision of results and rate of convergence, three sets of convergence tolerances are used:

(a) $\varepsilon_D = 10^{-3}, \quad \varepsilon_\lambda = 10^{-4}, \quad \varepsilon_f = 10^{-4},$

(b) $\varepsilon_D = 2 \cdot 10^{-5}, \quad \varepsilon_\lambda = 10^{-5}, \quad \varepsilon_f = 10^{-4},$

(c) $\varepsilon_D = 10^{-7}, \quad \varepsilon_\lambda = 10^{-7}, \quad \varepsilon_f = 10^{-8}.$

All examples have been run on an IBM-compatible micro-computer, equipped with 8086 and 8087 coprocessors, 10 MHz clock, 640 KB of RAM memory and 20 MB in hard disk.

5.3.1. *Cantilever beam with concentrated load.* A bidimensional model for a cantilever beam, with rectangular cross-section, thickness $2L/15$ (L is the beam length), made up of von Mises material and subjected to a concentrated load, has been discretized as shown in Fig. 1 by using 1080 constant strain triangles for plane stress and 1172 degrees of freedom. The approximate value 1.0024 M_Y/L (M_Y is the yield moment of the cross-section) for the limit load is computed in 12 iterations for the set of tolerances denoted (b). Elapsed time for the solving stage is 4690 seconds.

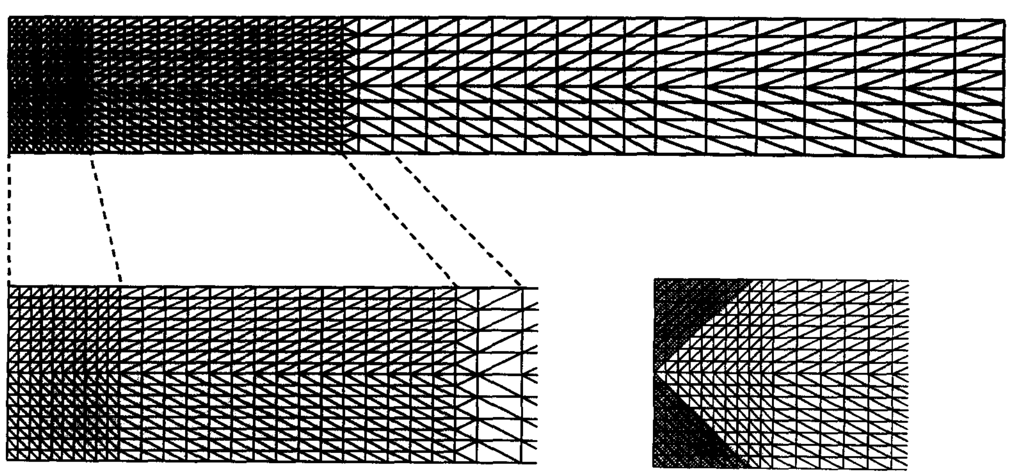

Fig. 1. Mesh and plastic regions for the bidimensional model of the cantilever beam. Discretization with 1080 elements and 1172 dof.

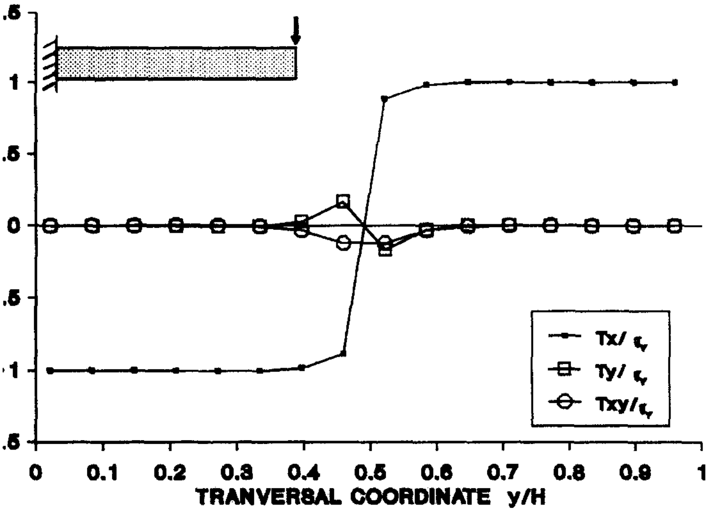

Fig. 2. Stresses along the clamped cross-section of the cantilever beam with concentrated load.

Figure 1 shows the plastic regions of the computed collapse mechanism. The distribution of stress components in the clamped cross-section is plotted in Fig. 2.

5.3.2. *Notched strip under traction.* This example is frequently used to test numerical methods in plasticity. It is a plate in plane stress with the dimensions defined in Fig. 3, which also shows the mesh containing 148 elements and 62 degrees of freedom.

Numerical results are shown in Table 2. The number of iterations is 8 (in 10 minutes) for the set of convergence tolerances (a), giving an approximate collapse load per unit of effective area:

$$2\alpha\bar{p} = 1.176410\sigma_Y. \tag{95}$$

For set (c) convergence is achieved in 14 iterations (in 18 minutes) and the approximate value is $1.176427\sigma_Y$.

5.3.3. *Thick walled tube under internal pressure.* This is a plane strain problem with symmetry of revolution. The symmetry is not exploited here in order to reduce the size of the model because we are interested in testing the efficiency of the proposed algorithm in large scale problems. So, a 90 degree sector of the cross-section is discretized with 448 triangles of linear velocity, as shown in Fig. 4, and 866 degrees of freedom.

In this case the Hessian of the plastic function is the matrix C corresponding to the plane strain given in Table 1, which is singular. Then, a different matrix, ${}^0C$, must be used

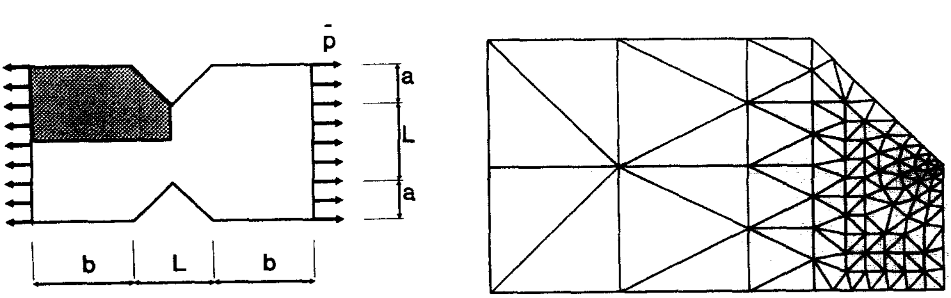

Fig. 3. Model for the notched strip, with 148 elements and 162 dof. Dimensions in the figure are $a = L/2$ and $b = 5L/4$.

Table 2. Limit loads for the notched strip under traction compared with values reported by Massonet (1979)

Author	$2\alpha\bar{p}/\sigma_Y$	dof	Remarks
Present	1.176	162	PC/XT-10′
Hill	1.155	—	Analytic
Yamada	1.124	290	IBM (7090) 10′
Nguyen, D.	1.192	170	IBM (360) 2′30
Nayak	1.186	178	ICT (1905E) 15′
Frey	1.180	178	IBM (360) 5′

in place of $\nabla^2 f$ for computing H. We adopt ${}^0\bar{C}$ as being C with a small perturbation (10^{-4} in this example) added to diagonal entries related to direct stresses.

The computed limit pressure $0.6939\sigma_Y$ is an upper bound of the theoretical value $0.6931\sigma_Y$. It is obtained in 8 iterations for the set of tolerances (a), and the elapsed time for the solving algorithm is 1300 seconds. Numerical and analytical stresses are plotted in Fig. 5.

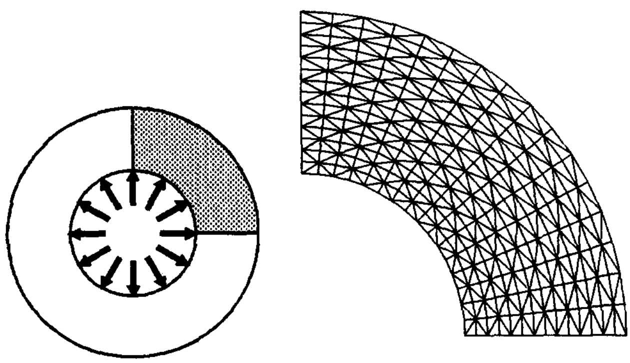

Fig. 4. Sector model for the thick-walled tube with external radius double that of the internal radius. Mesh contains 448 elements and 866 dof.

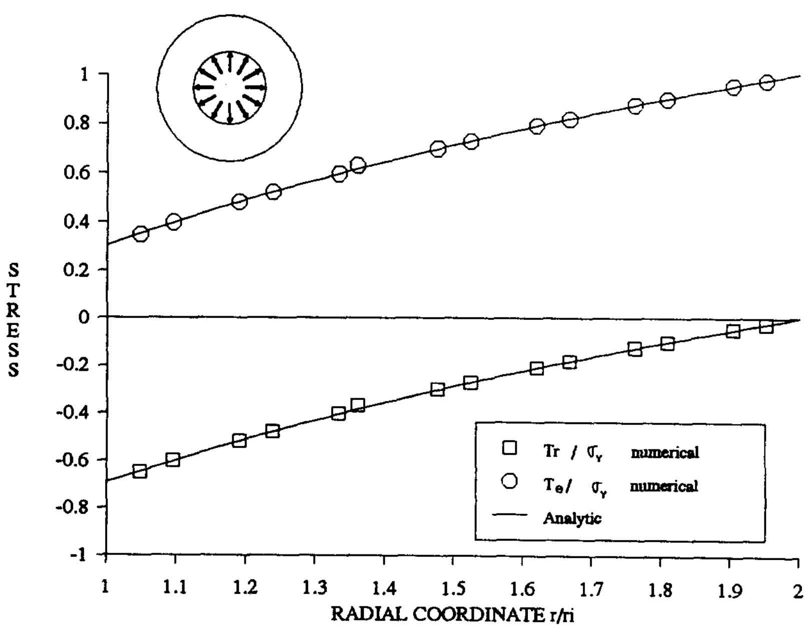

Fig. 5. Collapse stresses in the thick-walled tube.

5.3.4. *A case with non-unique solution.* The simple traction of a rectangular strip is used to test the behavior of the algorithm in problems where the collapse mechanism is nonunique.

No particular effects have been detected in this example related to the lack of uniqueness of the solution.

Figure 6 shows the undeformed mesh and a superposed one representing the collapse mechanism. Uniform deformation has been obtained for this mesh and also for other meshes with a concentrated band of small elements intended to induce localized deformations. Obviously the approximate collapse factor is exact in the present case.

5.3.5. *Representation of slip bands.* A square slab with a symmetrical internal slit subjected to traction presents localized deformation in the form of slip bands emanating from the roots of the crack. The presence of this effect in the approximate velocity field shown in Fig. 7 has not caused any trouble for the algorithm. The mesh has 475 degrees of freedom and the approximate collapse load per unit of effective area is

$$2\alpha\bar{p} = 1.0235560\sigma_Y. \tag{96}$$

The value $1.012\sigma_Y$ has been obtained by Gao Yang (1988) with a different finite element interpolation.

6. CONCLUSIONS

A new algorithm for limit analysis has been proposed in this paper, with the following features:

(i) It can be used to compute limit loads of a body whose plastic behavior is described by a set of linear or nonlinear yield functions.
(ii) It is based on a nonlinear programming method.
(iii) The structure of the limit analysis problem derived from a finite element discretization of the body has been taken into account in order that the algorithm should be able to solve large scale problems. In particular, a fictitious stiffness matrix is computed, and two linear systems are solved, in each iteration. These tasks can be performed in an element-by-element basis, using all well-known techniques to deal with large scale models.
(iv) Understanding and implementing the present algorithm does not require any special knowledge or software in mathematical programming; it only uses basic procedures

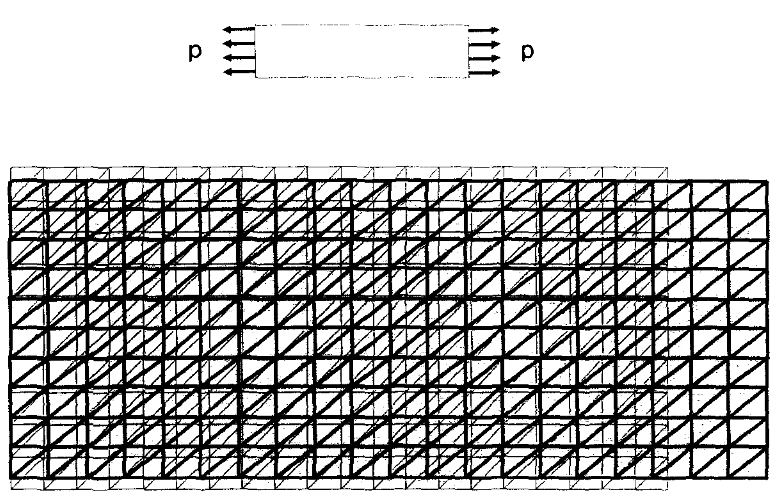

Fig. 6. Collapse deformation of a rectangular strip under traction.

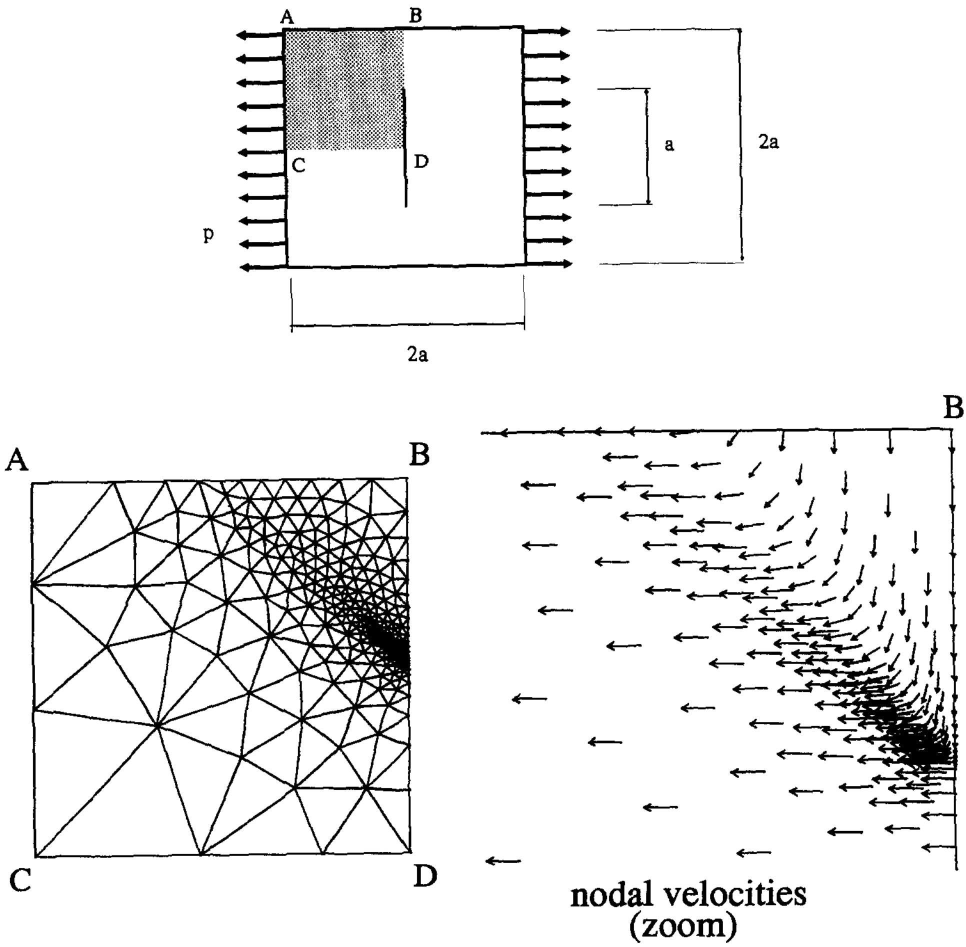

Fig. 7. Velocity field for the cracked square slab under traction (475 dof).

in structural analysis such as those for assembling element matrices and solving large scale linear systems with matrices similar to stiffness matrices.

(v) It shows a good convergence rate in the applications already performed, dealing with sizes ranging up to 1172 degrees of freedom. Moreover, the number of iterations does not increase too much with the model size.

(vi) Although the comparison of the performance of the present algorithm against linear programming methods is very dependent of the software used for the latter alternative, we can guess from our limited experience, that the nonlinear programming approach is the most promising one.

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APPENDIX

Proposition 1. The symmetric matrix

$$W = f_T^T H^{-1} f_T - \Lambda^{-1} G \tag{A1}$$

is positive definite.

Proof. Let us introduce the notation

$$A = f_T^T H^{-1} f_T, \tag{A2}$$

$$Z = -\Lambda^{-1} G = \mathrm{diag}\,(-f_j/\lambda_j) \tag{A3}$$

so that

$$W = A + Z. \tag{A4}$$

The term A is positive semi-definite because H is assumed positive definite, and the term Z is also positive semi-definite because all the components of f_j are negative or zero. In fact we do not allow f_j to be exactly zero, but it may converge to zero.

Hence W is at least positive semi-definite.

Moreover, to guarantee that A, and hence W, are definite, it suffices that $f_T\dot{\lambda}^*$ be different from zero whenever $\dot{\lambda}^*$ is not identically zero. Unfortunately, we cannot make this assumption on the plastic function f because sometimes the set of gradients f_{jT} corresponding to all plastic modes, also including those which are inactive, are linearly dependent. This situation occurs, for instance, with any pair of plastic modes representing opposite faces of the Tresca yield surface. On the other hand, the above mentioned hypothesis is true when stated only for admissible plastic factors, that is for any $\dot{\lambda}^*$ such that there exists some $T^* \in P$ satisfying

$$f_j(T^*)\dot{\lambda}_j^* = 0, \quad j = 1, \ldots, m. \tag{A5}$$

Then, if $\dot{\lambda}^*$ is admissible

$$\dot{\lambda}^* \neq 0 \quad \Rightarrow \quad f_T\dot{\lambda}^* \neq 0. \tag{A6}$$

This hypothesis is used next to prove that W is definite.

Consider any $\dot{\lambda}^*$ such that

$$W\dot{\lambda}^* \cdot \dot{\lambda}^* = 0. \tag{A7}$$

This implies that

$$H^{-1}(f_T\dot{\lambda}^*) \cdot (f_T\dot{\lambda}^*) = 0 \tag{A8}$$

and

$$\Sigma f_j(\dot{\lambda}_j^*)^2/\lambda_j = 0. \tag{A9}$$

The condition (A5) can be derived from (A9). Then $\dot{\lambda}^*$ must be identically zero because otherwise $f_T\dot{\lambda}^*$ should be different from zero, by virtue of (A6), and then (A8) could not be satisfied.

Proposition 2. The pseudo elastic–plastic matrix

$$\mathbb{D}^{ep} = H^{-1} - QW^{-1}Q^T \tag{A10}$$

is positive semi-definite; with W given by (45) and

$$Q = H^{-1} f_{\mathrm{T}}. \tag{A11}$$

Proof. In view of the fact that H is positive definite the pseudo elastic–plastic matrix can be decomposed as follows:

$$\mathbb{D}^{\mathrm{ep}} = H^{-1/2}(\mathrm{I} - V)H^{-1/2}, \tag{A12}$$

where I is the identity matrix and

$$V = \bar{Q}W^{-1}\bar{Q}^{\mathrm{T}}, \tag{A13}$$

$$\bar{Q} = H^{-1/2} f_{\mathrm{T}}, \tag{A14}$$

$$W = \bar{Q}\bar{Q}^{\mathrm{T}} - \Lambda^{-1}G. \tag{A15}$$

As a consequence of Proposition 1 the matrix V is positive semi-definite. Let μ be an eigenvalue of V, which must be nonnegative, and v an eigenvector related to μ. Then

$$(\bar{Q}W^{-1}\bar{Q}^{\mathrm{T}})v = \mu v. \tag{A16}$$

We multiply this equation by $\bar{Q}^{\mathrm{T}}$ and define

$$t = W^{-1}\bar{Q}^{\mathrm{T}}v \tag{A17}$$

to obtain

$$\bar{Q}^{\mathrm{T}}\bar{Q}t = \mu Wt. \tag{A18}$$

Substituting (A15), it follows that

$$(W - \Lambda^{-1}G) = \mu Wt, \tag{A19}$$

or equivalently

$$\Lambda^{-1}Gt = (\mu - 1)Wt. \tag{A20}$$

We deduce from (A16) that $\mu = 0$ whenever $t = 0$. We now assume that $t \neq 0$ and multiply the above equation by t. So

$$\Lambda^{-1}Gt \cdot t = (\mu - 1)Wt \cdot t. \tag{A21}$$

The left-hand side is nonpositive because $\Lambda^{-1}G$ is a diagonal matrix with nonpositive components, and the term $Wt \cdot t$ is positive by virtue of Proposition 1. Hence, the factor $(\mu - 1)$ is negative or zero. This result also holds true for $\mu = 0$.

Summarizing, the eigenvalues of V satisfy

$$0 \leqslant \mu \leqslant 1. \tag{A22}$$

This implies that $I - V$ is positive semi-definite, and by (A12) we conclude that $\mathbb{D}^{\mathrm{ep}}$ is also positive semi-definite.